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4 Optical Resonators

4.0 INTRODUCTION

Optical resonators, like their low-frequency, radio-frequency, and microwave counterparts, are used primarily in order to build up large field intensities with moderate power inputs. They consist in most cases of two, or more, curved mirrors that serve to "trap," by repeated reflections and refocusing, an optical beam that thus becomes the mode of the resonator. A universal measure of this property is the quality factor Q of the resonator. Q is defined by the relation

$$Q = \omega \times \frac{\text{field energy stored by resonator}}{\text{power dissipated by resonator}} \tag{4.0-1}$$

As an example, consider the case of a simple resonator formed by bouncing a plane TEM wave between two perfectly conducting planes of separation l so that the field inside is

$$e(z, t) = E \sin \omega t \sin kz \tag{4.0-2}$$

According to (1.3-22), the average electric energy stored in the resonator is

$$\mathscr{E}_{\text{electric}} = \frac{A\varepsilon}{2T} \int_0^l \int_0^T e^2(z, t)\, dz\, dt \tag{4.0-3}$$

where A is the cross-sectional area, ε is the dielectric constant, and $T = 2\pi/\omega$ is the period. Using (4.0-2) we obtain

$$\mathscr{E}_{\text{electric}} = \tfrac{1}{8}\varepsilon E^2 V \tag{4.0-4}$$

where $V = lA$ is the resonator volume. Since the average magnetic energy stored in a resonator is equal to the electric energy [1], the total stored energy is

$$\mathscr{E} = \tfrac{1}{4}\varepsilon E^2 V \tag{4.0-5}$$

Thus, recognizing that in steady state the input power is equal to the dissipated power, and designating the power input to the resonator by P, we obtain from (4.0-1)

$$Q = \frac{\omega \varepsilon E^2 V}{4P}$$

The peak field is given by

$$E = \sqrt{\frac{4QP}{\omega \varepsilon V}} \tag{4.0-6}$$

Mode Density in Optical Resonators

The main challenge in the optical frequency regime is to build resonators that possess a very small number, ideally only one, high Q modes in a given spectral region. The reason is that for a resonator to fulfill this condition, its dimensions need to be of the order of the wavelength.

Example: One-Dimensional Resonator

We consider the simple transverse electromagnetic (TEM) two-mirror resonator with a field distribution as given by Equation (4.0-2). The resonant frequencies are determined by requiring that the field vanish at $z = 0$ and at the location $z = L$ of the second reflector. This happens when

$$\sin k_m L = m\pi$$

$$m = 1, 2, \ldots$$

Using $k_m = \frac{\omega_m}{c} n$, where n is the index of refraction, we obtain $\omega_m = m(\pi c/nL)$ for the resonance frequencies corresponding to a frequency separation between adjacent modes of $\Delta\omega = \pi c/nL$. If we, arbitrarily, choose the criterion of sufficient mode spacing as $\Delta\omega = \omega$, we obtain $L = \lambda/2n$, i.e., the *linear dimension needs to be comparable to the wavelength* (in the medium).

Mode control in the optical regime would thus seem to require that we construct resonators with volume $\sim\lambda^3(\sim 10^{-12}\text{ cm}^3$ at $\lambda = 1\ \mu\text{m})$. This is not easily achievable. An alternative is to build large ($L \gg \lambda$) resonators but to use a geometry that endows only a small fraction of these modes with low losses (a high Q). In our two-mirror example, any mode that does not travel normally to the mirror will "walk off" after a few bounces and thus will possess a low Q factor. We will show later that when the resonator contains an amplifying (inverted population) medium, oscillation will occur preferentially at high Q modes, so that the strategy of modal discrimination by controlling Q is sensible. We shall also find that further modal discrimination is due to the fact that the atomic medium is capable of amplifying radiation only within a limited frequency region so that modes outside this region, even if possessing high Q, do not oscillate.

One question asked often is the following: Given a large ($L \gg \lambda$) optical resonator, how many of its modes will have their resonant frequencies in a given frequency interval, say, between ν and $\nu + \Delta\nu$? To answer this problem, consider a large, perfectly reflecting box resonator with sides, a,b,c along the x,y,z directions. Without going into modal details, it is sufficient for our purpose to take the amplitude field solution in the form

$$E(x,y,z) \propto \sin k_x x \sin k_y y \sin k_z z \tag{4.0-7}$$

(Resonators of different shapes will differ in detail, but for large, $L \gg \lambda$, resonators, the results are similar.)

$$k_x^2 + k_y^2 + k_z^2 = \left(\frac{\omega}{c} n\right)^2 \tag{4.0-8}$$

For the field to vanish at the boundaries, we thus need to satisfy

$$k_x = \frac{r\pi}{a}, \quad k_y = \frac{s\pi}{b}, \quad k_z = \frac{t\pi}{c} \qquad r,s,t \text{ any integers} \tag{4.0-8a}$$

With each such mode, we may thus associate a propagation vector $\mathbf{k} = \hat{x}k_x + \hat{y}k_y + \hat{z}k_z$. The triplet r,s,t defines a mode. Since replacing any integer with its negative does not, according to Equation (4.0-7), generate an independent mode, we will restrict, without loss of generality, r,s,t to positive integers. It is convenient to describe the modal distribution in $\mathbf{k}$ space, as in Figure 4-1. Since each (positive) triplet r,s,t generates an independent mode, we can associate with each mode an elemental volume in $\mathbf{k}$ space.

$$V_{\text{mode}} = \frac{\pi^3}{abc} = \frac{\pi^3}{V} \tag{4.0-9}$$

where V is the physical volume of the resonator. We recall that the length of the vector $\mathbf{k}$ satisfies Equation (4.0-8), rewritten here as

$$k(r,s,t) = \frac{2\pi\nu(r,s,t)}{c} n \tag{4.0-10}$$

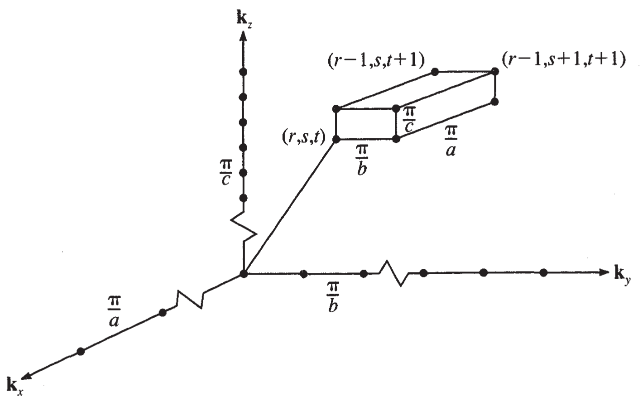

Figure 4-1 **k** space description of modes. Every positive triplet of integers r,s,t defines a unique mode. We can thus associate a primitive volume π^3/abc in **k** space with each mode.

To find the total number of modes with **k** values between 0 and k, we divide the corresponding volume in **k** space by the volume per mode:

$$N(k) = \frac{\left(\frac{1}{8}\right)\frac{4\pi}{3}k^3}{\frac{\pi^3}{V}} = \frac{k^3V}{6\pi^2}$$

(The factor 1/8 is due to the restriction of $r,s,t > 0$.)

We next use (4.0-10) to obtain the number of modes with resonant frequencies between 0 and ν:

$$N(\nu) = \frac{4\pi\nu^3n^3V}{3c^3}$$

The mode density, that is, the number of modes per unit ν near ν in a resonator with volume $V(\gg\lambda^3)$, is thus

$$p(\nu) = \frac{dN(\nu)}{d\nu} = \frac{8\pi\nu^2n^3V}{c^3} \tag{4.0-11}$$

where we multiplied the final result by 2 to account for the two independent orthogonally polarized modes that are associated with each r,s,t triplet.

The number of modes that fall within the interval $d\nu$ centered on ν is thus

$$N \simeq \frac{8\pi n^3\nu^2V}{c^3}\,d\nu \tag{4.0-12}$$

where V is the volume of the resonator. For the case of $V = 1\ \text{cm}^3$, $\nu = 3 \times 10^{14}$ Hz and $d\nu = 3 \times 10^{10}$, as an example, (4.0-12) yields $N \sim 2 \times 10^9$ modes. If the resonator were closed, all these modes would have similar values of Q. This situation

is to be avoided in the case of lasers, since it will cause the atoms to emit power (thus causing oscillation) into a large number of modes, which may differ in their frequencies as well as in their spatial characteristics.

This objection is overcome to a large extent by the use of open resonators, which consist essentially of a pair of opposing flat or curved reflectors. In such resonators the energy of the vast majority of the modes does not travel at right angles to the mirrors and will thus be lost in essentially a single traversal. These modes will consequently possess a very low Q. If the mirrors are curved, the few surviving modes will, as shown below, have their energy localized near the axis; thus the diffraction losses caused by the open sides can be made small compared with other loss mechanisms such as mirror transmission. (This point is considered in detail in Section 4.9. The subject of losses is also considered in Section 4.7.)

4.1 FABRY–PEROT ETALON

The Fabry–Perot etalon, or interferometer, named after its inventors [3], can be considered as the archetype of the optical resonator. It consists of a plane-parallel plate of thickness l and index n that is immersed in a medium of index n'.[1] Let a plane wave be incident on the etalon at an angle θ' to the normal, as shown in Figure 4-2(a). We can treat the problem of the transmission (and reflection) of the plane wave through the etalon by considering the infinite number of partial waves produced by reflections at the two end surfaces. The phase delay between two partial waves—which is attributable to one additional round trip—is given, according to Figure 4-2(a), by

$$\delta = \frac{4\pi nl\cos\theta}{\lambda} \tag{4.1-1}$$

where λ is the vacuum wavelength of the incident wave and θ is the internal angle of incidence. If the complex amplitude of the incident wave is taken as A_i, then the partial reflections, B_1, B_2, and so forth, are given by

$$B_1 = rA_i \qquad B_2 = tt'r'\,A_ie^{i\delta} \qquad B_3 = tt'r'^3\,A_ie^{2i\delta} \qquad \cdots$$

where r is the reflection coefficient (ratio of reflected to incident amplitude), t is the transmission coefficient for waves incident from n' toward n, and r' and t' are the corresponding quantities for waves traveling from n toward n'. The complex amplitude of the (total) reflected wave is $A_r = B_1 + B_2 + B_3 + \cdots$, or

$$A_r = \{r + tt'r'e^{i\delta}(1 + r'^2e^{i\delta} + r'^4e^{2i\delta} + \cdots)\}\,A_i \tag{4.1-2}$$

For the transmitted wave,

$$A_1 = tt'\,A_i \qquad A_2 = tt'r'^2e^{i\delta}\,A_i \qquad A_3 = tt'r'^4e^{2i\delta}\,A_i$$

[1] In practice, one often uses etalons made by spacing two partially reflecting mirrors a distance l apart so that $n = n' = 1$. Another common form of etalon is produced by grinding two plane-parallel (or curved) faces on a transparent solid and then evaporating a metallic or dielectric layer (or layers) on the surfaces.